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Dimension-reduction of FPK equation via equivalent drift coefficient

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Abstract The Fokker–Planck–Kolmogorov (FPK) equation plays an essential role in nonlinear stochastic dynamics. However, neither analytical nor numerical solution is available as yet to FPK equations for high-dimensional systems. In the present paper, the dimension reduction of FPK equation for systems excited by additive white noise is studied. In the proposed method, probability density evolution method (PDEM), in which a decoupled generalized density evolution equation is solved, is employed to reproduce the equivalent flux of probability for the marginalized FPK equation. A further step of constructing an equivalent coefficient finally completes the dimension-reduction of FPK equation. Examples are illustrated to verify the proposed method.
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Keywords FPK equation, drift coefficient, probability density evolution method, flux of probability, nonlinear systems

The Fokker–Planck–Kolmogorov (FPK) equation plays an essential role in many disciplines of science and engineering when white noise is involved in a dynamical system.[1] The solution of a FPK equation provides the transition or instantaneous joint probability density function of a state vector. Unfortunately, the analytical solution to FPK equations is only available for linear systems and some special nonlinear systems with natural boundary conditions.[1,2] Moreover, the dimension of FPK equation is identical to the dimension of the related Itô's stochastic differential equation, which is usually very large in practice. Despite great endeavors, the solution of FPK equation for generic high-dimensional nonlinear stochastic systems, no matter by analytical approaches or by numerical methods, is still a great challenge.[3,4] An alternative is to reduce the dimension of the FPK equation so that the problem will become much easier to solve. For instance, the state-space-split method was proposed by Er.[5]

In the past decade a family of probability density evolution method (PDEM) was developed and extensively studied.[6] In this method, the principle of preservation of probability is laid as the foundation of stochastic dynamics. Based on this principle, the existing traditional equations including the Dostupov–Pugachev equation, Liouville equation and FPK equation could be derived in a unified manner. Moreover, a new family of generalized density evolution equation (GDEE), which is completely decoupled, i.e., the dimension untied from the original dynamical system, is established.[7,8] Solving GDEE will provide probabilistic information of the physical quantities of

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concern. This method was applied successfully in various fields.[9–11]

The most fundamental idea embedded in PDEM is to decouple the governing partial differential equation in terms of joint probability density function (PDF) through random event description of the principle of preservation of probability. Whereas the FPK equation could be derived from the state space description, the equivalence between these two descriptions makes a bridge that could decouple FPK equation based on PDEM. This has been done by the introduction of equivalent probability flux.[12] In this letter, a further step is taken to construct an equivalent drift coefficient. Thus the original high-dimensional FPK equation is reduced to a one-dimensional FPK-like partial differential equation. Two numerical instances are analyzed to illustrate the proposed idea.

It is established that the principle of preservation of probability is a foundation in stochastic dynamics.[7,8] Based on this principle, all the existing traditional equations, e.g., the Dostupov–Pugachev equation, Liouville equation, and FPK equation, could be derived in a unified manner.[8] Moreover, a family of new GDEE could be established.[7]

Without loss of generality, a non-linear dynamical system with state equation is considered

$$\dot{\boldsymbol{X}} = \boldsymbol{A}(\boldsymbol{X},t) + \boldsymbol{B}(\boldsymbol{X},t)\boldsymbol{\xi}(t), \tag{1}$$

where $\boldsymbol{X} = (X_1, X_2, \cdots, X_n)^{\mathrm{T}}$ represents the n-dimensional state vector, $\boldsymbol{A} = (A_1, A_2, \cdots, A_n)^{\mathrm{T}}$ denotes the n-dimensional drift vector, $\boldsymbol{B} = [B_{ij}]_{n\times r}$ is the input matrix representing force influence, and $\boldsymbol{\xi}(t) = (\xi_1(t), \xi_2(t), \cdots, \xi_r(t))^{\mathrm{T}}$ is the r-dimensional white noise excitation vector with $E[\boldsymbol{\xi}(t)] = \boldsymbol{0}$ and $E[\boldsymbol{\xi}(t)\boldsymbol{\xi}^{\mathrm{T}}(t+\tau)] = \boldsymbol{D}\delta(\tau)$, where $\delta(\cdot)$ is Dirac's delta function.

The system of Eq. (1) can be considered as an Itô's stochastic differential equation. The joint PDF of $\boldsymbol{X}(t)$, denoted by $p_{\boldsymbol{X}}(\boldsymbol{x},t)$, is governed by the following FPK equation in this case[6]

$$\frac{\partial p_{\boldsymbol{X}}(\boldsymbol{x},t)}{\partial t} = -\sum_{j=1}^{n} \frac{\partial (A_j(\boldsymbol{x},t) p_{\boldsymbol{X}}(\boldsymbol{x},t))}{\partial x_j} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \sigma_{ij}(\boldsymbol{x},t) p_{\boldsymbol{X}}(\boldsymbol{x},t)}{\partial x_i \partial x_j}, \tag{2}$$

where σ_{ij} is the component of $\boldsymbol{\sigma}(\boldsymbol{x},t) = \boldsymbol{BDB}^{\mathrm{T}}$. It is noted that the above equation could also be deduced from the principle of preservation of probability.

Alternatively, the stochastic excitation vector $\boldsymbol{\xi}(t)$ can be presented by a random function vector, e.g., $\boldsymbol{\xi}(t) = \boldsymbol{F}(\boldsymbol{\Theta},t) = (F_1(\boldsymbol{\Theta},t), F_2(\boldsymbol{\Theta},t), \cdots, F_r(\boldsymbol{\Theta},t))^{\mathrm{T}}$, where $\boldsymbol{\Theta} = (\Theta_1, \Theta_2, \cdots, \Theta_s)$ are the involved basic random variable vector with known PDF $p_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$, in which $\boldsymbol{\theta} = (\theta_1, \theta_2, \cdots, \theta_s)$. This can be implemented by, say the spectral representation method[13] or the stochastic harmonic function representation for bandwidth-limited white noise processes.[14] By doing so, Eq. (1) can be rewritten as

$$\dot{\boldsymbol{X}} = \boldsymbol{G}(\boldsymbol{X},\boldsymbol{\Theta},t), \tag{3}$$

where $\boldsymbol{G}(\boldsymbol{X},\boldsymbol{\Theta},t) = \boldsymbol{A}(\boldsymbol{X},t) + \boldsymbol{B}(\boldsymbol{X},t)\boldsymbol{F}(\boldsymbol{\Theta},t)$.

From the random event description of the principle of preservation of probability, a GDEE could be obtained[6,7]

$$\frac{\partial p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)}{\partial t} = -\dot{X}_l(\boldsymbol{\theta},t)\frac{\partial p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)}{\partial x_l}, \tag{4}$$

where X_l is the l-th component of $\boldsymbol{X}$, and $p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)$ is the joint PDF of $(X_l(t),\boldsymbol{\Theta})$.

It is noted that the FPK equation is an n-dimensional partial differential equation with complex coefficients, and thus is usually unsolvable for high-dimensional systems.[4] In contrast, the dimension of GDEE could be arbitrary natural number, and only depends on how many physical quantities are of concern. Because both FPK equation and GDEE are based on the principle of preservation of probability, there is a possibility of bridging them to yield new results.

For the systems subjected to additive white noise excitations, i.e., $\boldsymbol{x}$ does not occur explicitly in $\boldsymbol{B}$ and $\boldsymbol{\sigma}$ and thus $\boldsymbol{B}(\boldsymbol{x},t)=\boldsymbol{B}(t)$ and $\boldsymbol{\sigma}(\boldsymbol{x},t)=\boldsymbol{\sigma}(t)$, Eq. (2) could be marginalized as by integrating on both sides in terms of $x_1,x_2,\cdots,x_n$ excluding x_l[12]

$$\frac{\partial p_{X_l}(x_l,t)}{\partial t} = -\frac{\partial J_{\mathrm{drift}}(x_l,t)}{\partial x_l} + \frac{\sigma_{ll}(t)}{2}\frac{\partial^2 p_{X_l}(x_l,t)}{\partial x_l^2}, \tag{5}$$

where $p_{X_l}(x_l,t)=\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}p_{\boldsymbol{X}}(\boldsymbol{x},t)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_l\,\mathrm{d}x_{l+1}\cdots\mathrm{d}x_n$, $J_{\mathrm{drift}}(x_l,t)$ is the probability flux contributed by drift, i.e.

$$J_{\mathrm{drift}}(x_l,t) = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}A_l(\boldsymbol{x},t)p_{\boldsymbol{X}}(\boldsymbol{x},t)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_l\,\mathrm{d}x_{l+1}\cdots\mathrm{d}x_n, \tag{6}$$

which could be further replaced by an equivalent flux due to drift $J^{\mathrm{eq}}_{\mathrm{drift}}(x_l,t)$ which is constructed by the solution of GDEE, i.e.

$$J^{\mathrm{eq}}_{\mathrm{drift}}(x_l,t) = \int_{\Omega_{\boldsymbol{\Theta}}}\Big(\dot{X}_l(\boldsymbol{\theta},t)-\sum_{k=1}^{r}B_{lk}F_k(\boldsymbol{\theta},t)\Big)p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)\,\mathrm{d}\boldsymbol{\theta}, \tag{7}$$

where $p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)$ satisfies GDEE in Eq. (4).

Equations (5) and (7) can be obtained by combining Eqs. (2) and (4) and considering the equivalence between the state space description and random event description. The results from Eq. (5) are expected to be superior to the direct result from Eq. (4), as indicated in Ref. 12.

In the present paper, a further step is proposed. Because $p_{X_l}(x_l,t)$ could also be obtained by solving Eq. (4) and then marginalizing the result by $p_{X_l}(x_l,t)=\int_{\Omega_{\boldsymbol{\Theta}}}p_{X_l\boldsymbol{\Theta}}(x_l,\boldsymbol{\theta},t)\,\mathrm{d}\boldsymbol{\theta}$. This could be taken as an intermittent result, denoted as $\hat{p}_{X_l}(x_l,t)$, then an equivalent drift coefficient could be constructed as

$$A^{\mathrm{eq}}_l(x_l,t) = J^{\mathrm{eq}}_{\mathrm{drift}}(x_l,t)/\hat{p}_{X_l}(x_l,t) \tag{8}$$

on the support of $\hat{p}_{X_l}(x_l,t)$. Substituting Eq. (8) into Eq. (5) immediately yields

$$\frac{\partial p_{X_l}(x_l,t)}{\partial t} = -\frac{\partial (A_l^{\mathrm{eq}}(x_l,t)p_{X_l}(x_l,t))}{\partial x_l} + \frac{\sigma_{ll}(t)}{2}\frac{\partial^2 p_{X_l}(x_l,t)}{\partial x_l^2}. \tag{9}$$

This is a one-dimensional FPK-like equation.

To reduce the error of numerical solution, the equivalent drift coefficient should be fitted according to intrinsic information of the dynamical systems. To this end, Eq. (4) is firstly solved, then Eqs. (7) and (8) are adopted to construct the equivalent drift coefficient. The coefficient could be smoothed before being substituted into Eq. (9), which is finally solved by, say the finite difference method with the Crank–Nicolson scheme in the present paper. Alternatively, because it is a one-dimensional equation, it could also be solved by various other approaches, e.g., the path integral method.[15]

A multi-degree-of-freedom (MDOF) linear system and a non-linear Duffing oscillator are presented to illustrate the proposed method.

Example 1 Consider a linear 9-story shear frame subjected to the ground motion which is modeled by a Gaussian white noise process. The lumped masses of each story are 9.78×10^4 kg, and the lateral inter-story stiffness are 9.9×10^7 N/m for the first and second story and 8.88×10^7 N/m for the rest 7 stories. Rayleigh damping is used, i.e., $\boldsymbol{C} = a\boldsymbol{M} + b\boldsymbol{K}$, where $a = 0.38$ and $b = 0.005$.

In this case, originally an 18-dimensional FPK equation should be resolved to get the probabilistic information of the state vector. As a linear system, the solution is known as a joint Gaussian distribution. If only the PDF of the top velocity is of concern, it is adequate to use the proposed method to solve a one-dimensional FPK-like equation. To this end, the spectral representation method is employed for the realization of bandwidth-truncated Gaussian white noise, and the Sobol' point set is adopted as the representative point set. The standard deviation of the seismic excitation is taken as 0.1 g while the relationship between the standard deviation and the intensity is $\sigma_0 = \sqrt{S\omega_{\mathrm{u}}/\pi}$, where ω_{u} is the cut-off upper circular frequency. $\Delta t = 0.0015$ s is taken as the time step. A total number of 1 000 representative time histories are adopted here.

The PDF of the top floor velocity at three different time instants by the proposed method, together with the exact solutions, which are Gaussian distribution in this case, are shown in Fig. 1. It is seen that by the proposed method high accuracy could be achieved even up to the order of magnitude of 10^{-6}.

Example 2 Consider a Duffing oscillator subjected to white noise

$$\ddot{\boldsymbol{X}} + \gamma\dot{\boldsymbol{X}} + \boldsymbol{X} + \varepsilon\boldsymbol{X}^3 = \boldsymbol{\xi}(t), \tag{10}$$

where ε is a nonlinear factor being 0.3. The intensity of Gaussian white noise is set as 0.5.

In this example originally a two-dimensional FPK equation is to be solved. The stationary solution is known. In the proposed method only a one-dimensional FPK-like equation is to be solved. For this purpose, the spectral representation method is again employed and the Sobol' point set is adopted as the representative point set. $\Delta t = 0.001$ s is taken as the time step. A

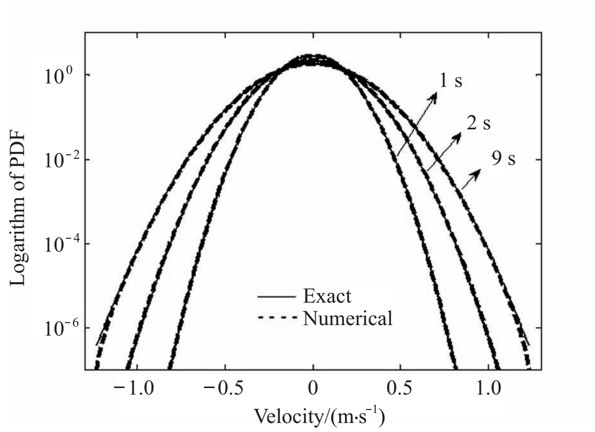


Fig. 1. PDF of the top velocity of the 9-story shear frame.

total number of 1 000 representative time histories are adopted in this example. For comparison, the Monte Carlo simulation with 2×10^5 samples is implemented to yield the mean and standard deviation, then Gaussian distributions with the obtained mean and standard deviation at three different time instants are plotted in Fig. 2 (labeled as "MCS"). The PDF of the velocity of the Duffing oscillator, together with the solutions at three time instants by Monte Carlo simulation, are shown in Fig. 2. Again it is seen that for this nonlinear system, the proposed approach performs well even at the order of magnitude of 10^{-5}. Actually, the PDF at 18 s is almost indistinguishable from the stationary solution which is not shown in Fig. 2 for clarity of the figure. Besides, the PDF evolution could also be obtained by the proposed method, as shown in Fig. 3, where the PDF surface and PDF contour are plotted against time. From Fig. 3, the process of evolving to a stationary state could be inspected.

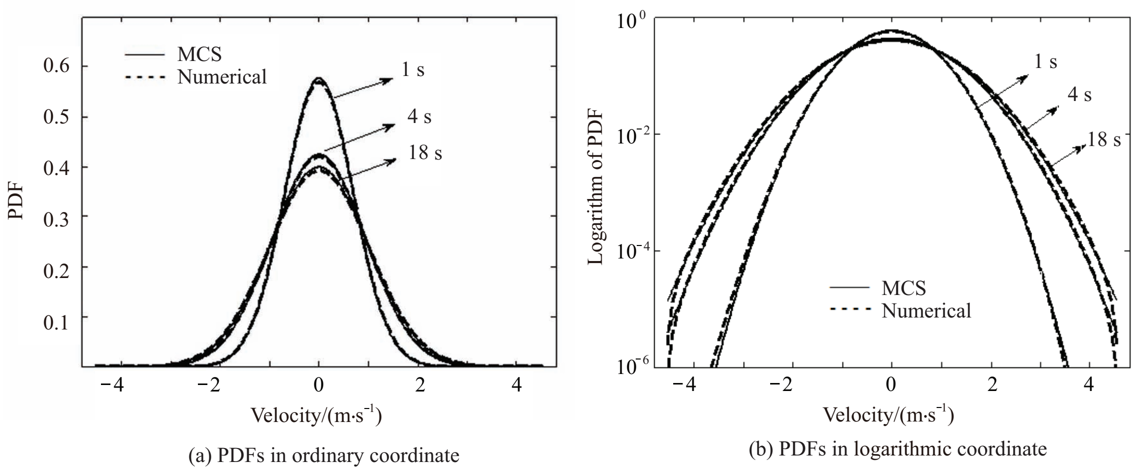
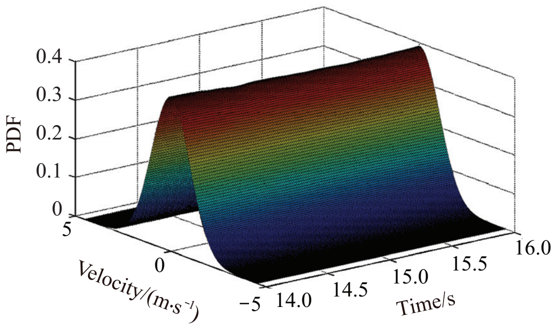
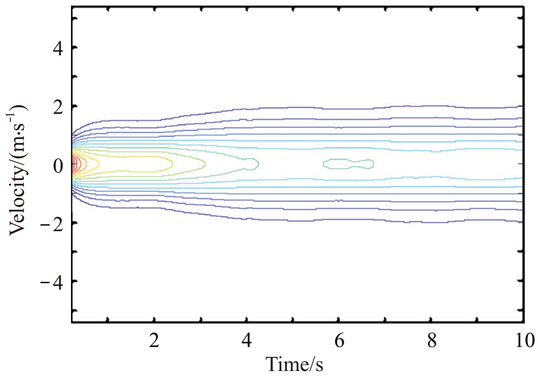


Fig. 2. Typical PDFs of the velocity of the Duffing oscillator.

The FPK equation plays an important role in science and engineering. However, the difficulty in solving high-dimensional FPK equations hinders its applications to high-dimensional nonlinear



(a) PDF surface



(b) PDF contour

Fig. 3. PDF evolution of the velocity of the Duffing oscillator.

systems. In the present paper, for systems excited by additive white noise, an equivalent drift coefficient is constructed based on the equivalence of probability flux, which could be obtained by PDEM in advance. Examples show that the proposed method is promising.

Problems to be further studied include (1) the applicability of the proposed approach to general nonlinear systems, and (2) more robust and efficient numerical methods for the solution of the equivalent FPK-like equations.

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